A Friedman-Savage Consumer Almost Gambles: A Continuous Time Model of Consumption and Investment with Non-Concave Utility

Byung Lim Koo∗, Hyeng Keun Koo†, Jung Lim Koo‡

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Abstract

We examine the consumption and portfolio decisions of an agent with Friedman-Savage type period utility in continuous time. We find the Friedman-Savage consumer does not gamble, but will aggressively invest in risky activities for wealth levels that support a minimum subsistence level of consumption. As the market premium of risk approaches zero, the agent becomes infinitely aggressive for these wealth levels. Also, there will be a jump in consumption corresponding to the region where the utility is convex. Last, the agent becomes more aggressive as patience decreases. (JEL D11, D91, G11)

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∗Wang Yanan Institute for Studies in Economics, Xiamen University, Xiamen, Fujian 361005, China. Email: koolimy@hotmail.com.
†Department of Financial Engineering, Ajou University, Suwon 443-749, Republic of Korea. Email: hkoo@ajou.ac.kr.
‡Wang Yanan Institute for Studies in Economics, Xiamen University, Xiamen, Fujian 361005, China. Email: koolimy@gmail.com.
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1 Introduction

Risk-taking behavior of individuals has long puzzled economists. Individuals often appear to take excessive risk, e.g., bet on sub-fair gambles, or invest in highly risky entrepreneurial projects. Simultaneously, they show conservative behavior, that is, they buy casualty and life insurance contracts. In an attempt to resolve this seemingly contradictory behavior, Friedman and Savage (1948) proposed a utility function which has convex as well as concave parts. An expected utility maximizer with such a utility function exhibits risk-loving behavior for a certain range of wealth and risk-averse behavior for the other wealth levels, and thus may gamble and buy insurance at the same time.

One can challenge, however, the intuition as myopic. The intuition is derived from a one-period model and does not consider multi-period optimization. In a multi-period setting economic agents typically smooth their behavior by trading across time and states. In the multi-period setting a Friedman-Savage type utility function may not generate the simultaneous attitude of risk-loving and risk aversion. In a general context Raiffa (1968) raised this concern by saying that “the utility function that a person works with today should be sensitive to the demands or investment opportunities that he perceives will be available to him in the future.”

The challenge is similar to the one raised for the Keynesian hypothesis about consumption which stipulates that consumption is a function of current income. Friedman’s (1957) permanent income hypothesis and Modigliani and Blumberg’s (1955) life cycle theory of consumption revised the hypothesis by saying that consumption is a result of lifetime planning. Thus consumption is a function of current income.

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1 The passage is re-quoted from Hakansson (1970).
and the present value of future income, not necessarily that of solely current income. A multi-period consideration therefore implies smoothing of consumption. Such smoothing may apply to the risk-taking of a Friedman-Savage utility maximizer over her lifetime as well. Indeed Bailey et al. (1980) and Hartley and Farrell (2002) show that an economic agent with such a utility function does not necessarily gamble in a two-period example.

In this paper we study the lifetime consumption and investment optimization problem of an infinitely-lived Friedman-Savage utility maximizer in continuous time. The questions we ask are the following. Is the agent willing to gamble? More generally, what is her behavior toward risk? What is her consumption behavior? Our answer to the first question is both negative and affirmative. Negatively, the agent exhibits risk aversion for all levels of wealth and thus is not willing to take a fair gamble. The intuition is as follows. The agent optimally consumes only at the extreme points of the non-concave region of the utility function and not at any point inside the non-concave region. This is equivalent to the agent using the concave hull of her utility function for optimization. Although using a concave hull as a utility function is effectively the result of a fair gamble in a one-period model, in a continuous time model the existence of a continuum of time points and saving and investment opportunities enables the agent to use the concave hull without taking a gamble. Our result is consistent with Bailey et al.’s (1980) finding in their two-period model in the sense that the agent can use her intertemporal trade to replicate or dominate the result of a gamble. On the other hand, Hartley and Farrell (2002) have claimed that a gamble can persist even in an infinite horizon model. In our continuous-time model, however, demand for gambling in its strict sense does not arise because of intertemporal smoothing of risk.

On the affirmative side, we show that for low levels of wealth the agent becomes infinitely aggressive in risk taking as the market premium of risk approaches zero. Specifically, a Friedman-Savage type agent is very aggressive when she consumes at a level corresponding to the left-end point of the convex region of the utility function. We further find that the agent adjusts her consumption to the right-end point of the region very soon. Thus, in our model the agent with a Friedman-Savage utility function does not gamble, but *almost gambles* when wealth is low.
This finding positively confirms Friedman and Savage’s original intuition.

Generally, the agent exhibits aggressive risk-taking behavior for low levels of wealth. These results are similar to those of Vereschagina and Hopenhayn (2009), who found that poor entrepreneurs take more risk. Furthermore, the agent’s risk taking attitude for these wealth levels is independent of her risk taking attitude for large wealth levels. The result is similar to Dybvig’s (1995) result that the risk aversion of an agent is independent of her endowed utility function if she cannot tolerate decline of consumption over time.

Analysis of the optimal consumption behavior of the Friedman-Savage consumer shows that there is a jump in consumption for levels of wealth that correspond to the convex region of the utility function. For low levels of wealth in this region, consumption will stay at a minimum subsistence level even though wealth increases. However, once the agent’s wealth hits a certain trigger level her consumption will jump to a higher level $\bar{C}$.

In contrast to Bailey et al. (1980) and Hartley and Farrell (2002) which deal with risk taking exogenously we determine the risk-taking behavior endogenously as a selection of investment portfolios. We study portfolio choice in an economy where there exist risky as well as risk-free investment vehicles. For this purpose we use the martingale approach to consumption and investment problems proposed by Karatzas et al. (1987) and Cox and Huang (1989). This approach transforms the dynamic wealth evolution equation into a static budget constraint, which enables one to solve the dual problem of the consumer’s choice. The resulting Hamilton-Jacobi-Bellman equation is linear and admits a simple closed-form solution. We derive optimal consumption and optimal portfolio of investments in closed form from the dual value function.

There is extensive literature on Friedman-Savage type utility functions. We mention only a few among the literature. Markowitz (1952) proposes a revision of the Friedman-Savage utility function, where the utility function has multiple convex and concave regions. Kwang (1965) shows that simultaneous gambling and insurance can arise due to indivisibility of expenditure. The resulting value function will have a convex region. Assuming a model with positive transaction costs, Flemming (1969) creates a utility function of windfalls that shares many common
features with the utility function proposed by Markowitz (1952). Specifically, there are some non-concave regions in the utility function due to “dents” in the utility function. Hakansson (1970) finds that borrowing constraints can give rise to a Friedman-Savage type value function even though utility of consumption and induced future preferences display diminishing marginal utility. Applebaum and Katz (1981) show that capital market imperfections may give rise to a Friedman-Savage type value function. Brunnermeier (2004) shows that the necessity of learning one’s optimal consumption bundle and bounded rationality can give rise to a Friedman-Savage type utility function.

The literature studying portfolio selection using partially convex utility functions have generally focused on maximizing utility of terminal wealth when the agent has loss aversion (see Berkelaar et al. (2004), Gomes (2005), Jin and Zhou (2008), He and Zhou (2010), Zhang et al. (2011), and Jin and Zhou (2012)). However, such models bear a few limitations. As Campbell and Viceira (1999) criticize, first, such behavior may not be applicable to most investment activities except for special cases such as managing retirement funds. Second, consumption is a critical economic choice that reflects an agent’s risk taking, and therefore neglecting consumption makes it difficult to relate the research to the literature which investigate the economic consequences of risk taking, such as asset pricing (e.g., Merton (1973), Breeden (1979), Mehra and Prescott (1985), Campbell and Cochrane (1999)).

The paper is organized as follows. In section 2 we specify the continuous time consumption and portfolio optimization problem. In Section 3 we solve the model and give closed form solutions for the value function and optimal consumption and investment policies. Section 4 contains an example of a modified CRRA utility function. Section 5 concludes.

2 The Model

We consider an infinitely-lived economic agent who optimizes utility of consumption over her lifetime. The agent’s preference is characterized by the following von Neumann-Morgenstern utility function

\footnote{We owe the expression “dents” to Flemming (1969).}
Figure 1: Friedman-Savage type felicity function

\[ E \left[ \int_0^\infty e^{-\rho t} u(C_t) \, dt \right], \]

where the felicity function \( u(C) \) is strictly increasing and given by the function of Friedman-Savage type shown above.

We see that this utility function is concave for levels of consumption above \( \bar{C} \), but convex for lower levels of consumption \( C \leq \bar{C} \). Friedman and Savage’s original intuition behind this utility was to explain the co-existence of gambling and insurance. If one is consuming at \( C > \bar{C} \), she will not wager her current consumption for a lottery where she might end up with lower utility than her current utility. If one is consuming at \( C \leq C < \bar{C} \), however, she has an incentive to bet on even a subfair gamble, since by doing so she will be able to achieve a higher utility level than her current one.

Without loss of generality we will focus on a situation where \( C = 0 \). \( C \) can be regarded either as a current benchmark or as a minimum subsistence level of consumption.

We further make two assumptions. First, for simplicity we assume that the utility at the minimum subsistence level of consumption is zero. Second, we will assume that in the concave region of the utility function, the function is strictly concave.

There are two assets to be invested in, a risk-free asset and a risky asset. The risky asset need not be limited to stocks traded in an exchange, but can include any risky investment activity, such as entrepreneurship. Though we consider only one
risky asset for simplicity, one can extend this model to a situation with multiple risky assets without much difficulty. The return on the risk-free asset is constant $r$. Namely, the price $S_0(t)$ of the risk-free asset evolves according to the following equation

$$\frac{dS_0(t)}{S_0(t)} = r dt, \quad S_0(0) = s.$$ 

The return of the risky asset is normally distributed over a small time interval $(t, t + dt)$ with mean equal to $\mu dt$ ($\mu > r$) and standard deviation equal to $\sigma \sqrt{dt}$. Namely, the price $S_1(t)$ of the risky asset satisfies the following stochastic evolution equation

$$\frac{dS_1(t)}{S_1(t)} = \mu dt + \sigma dW(t),$$

where $W(t)$ is a standard Wiener process defined on an appropriate probability space which will be denoted by $(\Omega, \mathcal{F}, \mathbb{P})$. $\mathbb{P}$ is the probability measure and $\mathcal{F}$ is the filtration generated by the Wiener process augmented by the null sets of $\mathbb{P}$.

According to the above, the agent faces a constant investment opportunity: the interest rate and the mean and standard deviation of the return on the risky asset are assumed to be constant. We make the assumption for parsimony of the model in order to focus on the essence of the consequences of the Friedman-Savage felicity function.

Let $C_t$ be the rate of consumption chosen by the agent at time $t$, and $\pi_t$ be the dollar amount of investment in the risky asset at time $t$. Then we state the following dynamic budget constraint:

$$dX(t) = (rX(t) - C(t) + (\mu - r) \pi(t)) dt + \sigma \pi(t) dW(t), \quad (1)$$

where

$$X(0) = x,$$

$$\lim_{t \to \infty} X(t) \geq 0.$$ 

The above equation says that the agent’s wealth earns interest at the rate of $r$, appreciates at the rate equal to the risk premium of the risky asset proportional to her investment, and decreases by her consumption. She also faces risk proportional to
her investment and the standard deviation of the risky asset’s return. The condition \( \lim_{t \to \infty} X(t) \geq 0 \) guarantees that the agent eventually pays off all her debt and is called the transversality condition.

We assume a frictionless financial market, or in other words, a market with no transaction costs, no taxes, and no short-selling restrictions. Since there exists one source of risk, one risky asset driven by the risk, and no friction in the market, the financial market is complete. Thus we will be able to apply the well-developed machinery for a complete market (see e.g. Cox and Huang (1989)) to our model.

Now we state the optimization problem of the economic agent.

Problem A  \[ \max_{c,X} E \left[ \int_0^\infty e^{-\rho t} u(C_t) \, dt \right] \]
subject to
\[ dX(t) = \left( rX(t) - C(t) + (\mu - r) \pi(t) \right) dt + \sigma \pi(t) dW(t), \]
\[ X(0) = x, \]
and
\[ \lim_{t \to \infty} X(t) \geq 0. \]

Note that the dynamic budget constraint is equivalent to the following static budget constraint:
\[ E \left[ \int_0^\infty H(t) C(t) \, dt \right] \leq x, \]  
where \( H(t) = e^{-\left( r + \frac{1}{2} \theta^2 \right) t - \theta W(t)} \) is the state price density, and \( \theta = \frac{\mu - r}{\sigma} \) is the market price of risk.

Karatzas et al. (1987), Cox and Huang (1989), Dybvig and Huang (1989), and Dybvig (1995) developed the above transformation of the dynamic wealth evolution equation into a static budget constraint. The static budget constraint has the following intuitive meaning: the present discounted value of future consumption streams should be less than or equal to initial wealth. Here the existence of risk implies one discounts future consumption \( C(t) \) not with interest rate \( r \) but by the
The stochastic discount factor $H(t)$. The dynamic budget constraint satisfies the static budget constraint even in an incomplete market (He and Pearson (1991)). The converse, however, is valid only in a complete market. That is, generally only in a complete market a consumption stream that satisfies the static budget constraint can be financed by trading assets so that the dynamic budget constraint is satisfied.

The equivalence of the dynamic and static budget constraints allows one to solve Problem A by using a standard Lagrangian. One chooses optimal $C(t)$ for every state $\omega$ and time $t$ and then considers a fictitious derivative asset which pays a stream of state contingent consumption at the rate of $C(t)$ at $t$. This asset can be bought with the agent’s initial wealth. Then, the agent’s portfolio is constructed to replicate the derivative by a standard option pricing technique (see e.g. Black and Scholes (1973), Campbell and Viceira (2002)). We will proceed in this way in the next section.

3 Solution to the model

We first consider the concave hull $\bar{u}(C)$ of $u(C)$ defined as follows:

$$\bar{u} = \inf_v \{ v : v \geq u \forall C \in \mathbb{R}, v \text{ is concave} \}.$$

For the utility function given in Figure 1, $\bar{u}$ is illustrated in the following figure:

![Figure 2: Concave hull of Friedman-Savage type felicity function](image)

The concave hull was first considered by Friedman and Savage. It represents the possible utility values when the agent bets on fair gambles in a one-period model.
Our objective in this section is to show that one can replicate the concave hull not by fair gambles but by intertemporal trade (i.e. by trading assets) in our continuous time model. Namely, we will show in the next theorem that in her optimization problem the agent effectively replaces her cardinal utility function by its concave hull.

For this purpose we consider the following modification of problem A:

Problem B  \[\begin{align*}
\max_{c,\pi} & \quad E \int_0^\infty e^{-\rho t} \bar{u}(C_t) \, dt \\
\text{subject to} & \\
& \frac{dX(t)}{dt} = (rX(t) - C(t) + (\mu - r) \pi(t)) \, dt + \sigma \pi(t) dW(t), \\
& X(0) = x, \\
\text{and} & \\
& \lim_{t \to \infty} X(t) \geq 0.
\end{align*}\]

We make the following standing assumption in order to make the problem well-posed:

Standing Assumption: Problem B is well-posed, i.e., has a finite maximum value.

We will now show that the maximized value of Problem B is indeed equal to that of Problem A, and one can choose an optimal policy for Problem B which is also optimal for Problem A.

A solution to Problem B is found by considering the following Lagrangian which uses the static budget constraint (2)

\[\begin{align*}
\mathcal{L} & = E \left[ \int_0^\infty e^{-\rho t} \bar{u}(C(t)) \, dt \right] + \lambda \left[ x - E \left[ \int_0^\infty H(t)C(t) \, dt \right] \right] \\
& = E \left[ \int_0^\infty e^{-\rho t} \left( \bar{u}(C(t)) - \lambda e^{\rho t} H(t)C(t) \right) \, dt \right] + \lambda x.
\end{align*}\]
The Lagrangian can be maximized at each state \( \omega \in \Omega \) at time \( t \). The maximization problem at state \( \omega \) and time \( t \) can be stated as follows:

\[
\text{Problem C} \quad \max_c \bar{u}(C(t)) - y(t)C(t) \quad \text{where} \quad y(t) = \lambda e^{pt}H(t).
\]

As mentioned in the previous section, because we are operating in a complete market setting the static budget constraint is equivalent to the dynamic budget constraint. Therefore when optimizing the objective function subject to the static budget constraint, if \( C(t) \) is chosen optimally then the optimal portfolio of investments is determined by the replicating portfolio of the fictitious derivative asset that pays a stream of contingent consumption at the rate \( C(t) \). Thus it suffices to optimize for \( C(t) \) only.

The first order condition for Problem C takes the following form

\[
y(t) \leq l : C^*(t) = \bar{u}^{-1}(y(t)), \quad (3)
\]

\[
l < y(t) \leq m : \quad C^*(t) = \bar{C}, \quad (4)
\]

\[
y(t) > m : \quad C^*(t) = 0. \quad (5)
\]

We denote the slope of the tangent of \( \bar{u}(C) \) at the point \( \bar{C} \) by \( l \), and the slope between 0 and \( \bar{C} \) as \( m \). If \( y(t) = m \), then \( C^*(t) \) can be chosen as any \( C \) between 0 and \( \bar{C} \). However, we choose \( \bar{C} \) in order to obtain an optimal policy which is optimal also for Problem A. \( ^3 \) Then our optimal policy will be \( (C^*(t))_{t=0}^{\infty} \) which satisfies the static budget constraint.

Here \( y \) is the dual variable which is equivalent to the marginal utility of wealth. In the first region, which corresponds to the concave region of \( \bar{u} \), \( \bar{u} \) is equal to the original utility function, and the optimal consumption policy is chosen using the inverse function of the first derivative of the original utility \( u \). The second first

\[ ^3 \text{We may also choose } C^*(t) = 0. \]
order condition implies the marginal utility of consumption is an element of the differential of \( \bar{u} \) at the point \( \bar{C} \), and thus because of the many-to-one correspondence the optimal consumption policy is chosen as \( \bar{C} \). By a similar argument, in the third region \( C^*(t) \) is chosen to be 0. Note that the first order conditions indicate that consumption would not be smooth and there would be a jump in consumption as the marginal utility of wealth decreases to less than \( m \).

By our choice \( \bar{u}(C^*(t)) = u(C^*(t)) \), thus \( C^*(t) \) is also an optimal policy for Problem B. We state the result as the following theorem.\(^4\)

**Theorem 1.** The maximized values of Problem A and B are equal and the consumption choice stated in the above first order condition and satisfying the static budget constraint is optimal for both problems.

**Proof**

Since \( \bar{u} \geq u \), the optimized value \( V_B \geq V_A \) where \( V_A \) (\( V_B \)) is the optimized value for Problem A (Problem B). But

\[
V_B = E \left[ \int_0^\infty e^{-\rho t} \bar{u}(C^*(t)) \, dt \right] = E \left[ \int_0^\infty e^{-\rho t} u(C^*(t)) \, dt \right] \leq V_A.
\]

Therefore \( V_A = V_B \) and \( (C^*(t)) \) is optimal for both Problem A and B. \( \blacksquare \)

Because of Theorem 1, we see that the shape of the convex region of the original utility function does not matter when solving the optimization problem.

Let us proceed to find the solution of the problem. We will state the following theorem which gives the agent’s value function and optimal consumption and portfolio. The proof uses dual formulation of the problem and will be given in Appendix.

**Theorem 2.** (1) The value function of the agent is equal to

\[
V(X) = \min_y (J(y) + yX) = J(J^{-1}(-X)) + J^{-1}(-X)X.
\]

\(^4\)We thank Philip H. Dybvig for pointing out the validity of the theorem.
where $J(y)$ is given by the following form

\begin{align*}
y \leq l : & \quad J = B_1 y^{\lambda_+} + J_{p,\bar{u}}(y), \\
l < y < m : & \quad J = A_2 y^{\lambda_-} + B_2 y^{\lambda_+} + \frac{\xi}{r} y + \frac{1}{\rho} u(\bar{C}), \\
m < y : & \quad J = A_3 y^{\lambda_-},
\end{align*}

and

\begin{align*}
\lambda_+ &= \frac{- (\rho - r - \frac{1}{2} \theta^2) + \sqrt{(\rho - r - \frac{1}{2} \theta^2)^2 + 2 \rho \theta^2}}{\theta^2}, \\
\lambda_- &= \frac{- (\rho - r - \frac{1}{2} \theta^2) - \sqrt{(\rho - r - \frac{1}{2} \theta^2)^2 + 2 \rho \theta^2}}{\theta^2}.
\end{align*}

$B_1, A_2, B_2, A_3$ satisfy the following smooth pasting conditions:

\begin{align*}
A_2 l^{\lambda_-} + B_2 l^{\lambda_+} - \frac{\bar{C}}{r} l + \frac{1}{\rho} u(\bar{C}) &= B_1 l^{\lambda_+} + J_{p,\bar{u}}(l), \\
\lambda_- A_2 l^{\lambda_- - 1} + \lambda_+ B_2 l^{\lambda_+ - 1} - \frac{\bar{C}}{r} &= \lambda_+ B_1 l^{\lambda_- - 1} + J'_{p,\bar{u}}(l), \\
A_2 m^{\lambda_-} + B_2 m^{\lambda_+} - \frac{\bar{C}}{r} m + \frac{1}{\rho} u(\bar{C}) &= A_3 m^{\lambda_-}, \\
\lambda_- A_2 m^{\lambda_- - 1} + \lambda_+ B_2 m^{\lambda_+ - 1} - \frac{\bar{C}}{r} &= \lambda_- A_3 m^{\lambda_- - 1}.
\end{align*}

(2) Optimal consumption is given by

\begin{align*}
C^*(t) &= \bar{u}^{-1}(y(t)) \quad y(t) \leq l, \\
C^*(t) &= \bar{C} \quad l < y(t) \leq m, \\
C^*(t) &= 0 \quad y(t) > m.
\end{align*}
(3) Optimal investment in the risky asset is given by

\[ \pi^*(t) = \frac{\theta y(t) J''(y(t))}{\sigma J'(y(t))} X(t). \]

(4) When optimal consumption is equal to 0 \((y > m)\),

\[ \frac{\pi^*}{\lambda} = -\frac{\theta (\lambda - 1)}{\sigma}. \]

Proof See Appendix.

In Theorem 2, \( J \) is called the dual value function and \( y \) is the agent’s marginal utility of wealth. The first equation in the theorem is a consequence of the minimax theory (see chapter 7 of Rockafellar (1970) or Karatzas and Wang (2001)). Optimal consumption is a result of the first order conditions in Equations (3) to (5). The agent’s optimal investment in the risky asset is proportional to \( \frac{y(t)J''(y(t))}{J'(y(t))} \) which can be regarded as the reciprocal of the relative risk aversion implied by the dual value function.

Theorem 2(4) states that when wealth is supporting a minimum subsistence level of consumption, the optimal proportion of investment in the risky asset is constant and independent of the shape of the strictly concave region. That is, the agent’s risk taking attitude at low wealth levels is independent of her attitude toward risk when her level of wealth is high. A similar feature occurs when an agent shows an extreme form of habit formation (Dybvig 1995).

As a consequence of Theorem 2 we now state our main result in the following theorem:

Theorem 3. (1) The value function is strictly concave.
(2) When optimal consumption is equal to 0, as the market risk premium approaches 0, the optimal proportion of investment in the risky asset approaches infinity

\[
\lim_{\theta \to 0} \frac{\pi^*}{X} = \infty,
\]

and the minimum wealth level which supports \( \bar{C} \) approaches 0.

Proof See Appendix.

Theorem 3(1) says that there is no demand for gambling. The value function is strictly concave, implying that the agent exhibits risk aversion. Thus there is no incentive for the agent to take a fair gamble. This is a feature of our continuous time model. Even though the period utility function has a convex region, intertemporal consideration smoothes her risk taking and as a result she does not exhibit risk-loving behavior.

Theorem 3(2) says that the optimal proportion of wealth in the risky asset approaches infinity while the threshold wealth level at which consumption jumps to \( \bar{C} \) approaches 0 as the market premium approaches zero. We are interested in this case because the outcome of investing in a risky asset with nearly zero risk premium is approximately equivalent to participating in a fair gamble. This result shows that when the risk premium of the risky asset is small, a Friedman-Savage type agent is very aggressive when wealth is low, and adjusts consumption to \( \bar{C} \) at a very low level of wealth. This behavior can be called *almost gambling*. Thus, in our continuous-time model the agent with a Friedman-Savage utility function does not gamble, but *almost gambles* at wealth levels which can support only minimum subsistence consumption.

Note also that Theorem 3(2) indicates that when \( \theta \) is close to 0 and when the agent is consuming at the minimum subsistence level, a decrease in \( \theta \) would make risk taking increase. However, when \( \theta \) is sufficiently large, an increase in \( \theta \) may
increase the proportion invested in the risky asset, because the higher market price of risk will make the risky assets more attractive. We show an example in the next section that demonstrates this property.

Proposition 1. For $y > m$, the change in the optimal proportion of investment in the risky asset with respect to changes in the agent’s subjective discount factor is greater than zero, i.e.,

$$\frac{\partial \pi}{\partial \rho} > 0.$$  

This implies that when the wealth is supporting only a minimum subsistence level of consumption, an impatient investor would invest a larger proportion of her wealth into the risky asset, while a patient investor would invest less. An impatient investor would prefer a higher level of consumption sooner rather than later, and thus would invest heavily in the risky asset with positive risk premium in an attempt to quickly increase her wealth to a level that would support a higher level of consumption.

### 4 An Example: a modified CRRA utility

Let us consider the following felicity function:

When $C \geq \bar{C}$,

$$u(C) = \frac{C^{1-\gamma}}{1-\gamma} + \eta, \; \gamma \neq 1.$$  

For $C < \bar{C}$, $u$ is convex as shown in the figure above. Here $\eta$ is such that the $u(C) = u(0) = 0$.

We assume that $r + \frac{\rho - r}{\gamma} + \frac{1}{2} \frac{\gamma - 1}{\gamma} \theta^2 > 0$, which guarantees problem B to be well-
Figure 3: Modified CRRA felicity function with Friedman-Savage shape

\[ J(y) \] takes the following form in this case:

\[
\begin{align*}
    y \leq l : & \quad J = B_1 y^{\lambda_+} + \kappa \gamma y^{1-\gamma} y^{1 - \frac{1}{\gamma}} + \frac{(u(\bar{c}) - \bar{c}^{1-\gamma})}{\rho}, \\
    l < y < m : & \quad J = A_2 y^{\lambda_-} + B_2 y^{\lambda_-} - \frac{\xi}{\gamma} y + \frac{1}{\rho} u(\bar{c}), \\
    m < y : & \quad J = A_3 y^{\lambda_-}.
\end{align*}
\]

where \( \kappa = \frac{\gamma}{\gamma + 1 - \frac{1}{\gamma} - \frac{\xi}{\gamma} - \frac{1}{\rho} - \frac{1}{\rho} - \frac{1}{\rho}} \), and the coefficients \( A_2, A_3, B_1, B_2 \) are derived from the smooth pasting conditions (6) to (9). \(^6\)

The optimal portfolio process is

\[
\begin{align*}
    y \leq l : & \quad \pi = \frac{\theta}{\sigma} \left( B_1 y^{\lambda_+} \left( \lambda_+ - 1 \right) y^{\lambda_+ - 1} + \kappa \gamma y^{1 - \frac{1}{\gamma}} \right), \\
    l < y < m : & \quad \pi = \frac{\theta}{\sigma} \left( A_2 y^{\lambda_-} \left( \lambda_- - 1 \right) y^{\lambda_- - 1} + \lambda_+ \left( \lambda_+ - 1 \right) B_2 y^{\lambda_- - 1} \right), \\
    m < y : & \quad \pi = \frac{\theta}{\sigma} \left( A_3 y^{\lambda_-} \left( \lambda_- - 1 \right) y^{\lambda_- - 1} \right).
\end{align*}
\]

\(^6\) See Appendix.
The optimal ratio of wealth in the risky asset is

\[ y \leq l : \quad \frac{\pi}{X} = -\frac{\theta}{\sigma} \left( \frac{B_1 \lambda_+ (\lambda_+ - 1) y^{\lambda+1-1} + \kappa y^{-\frac{1}{\gamma}}} {B_1 \lambda_+ y^{\lambda+1-1} - \kappa y^{-\frac{1}{\gamma}}} \right), \]

\[ l < y < m : \quad \frac{\pi}{X} = -\frac{\theta}{\sigma} \left( \frac{A_2 \lambda_- (\lambda_- - 1) y^{\lambda-1-1} + \lambda_- (\lambda_- - 1) B_2 y^{\lambda-1}} {A_2 \lambda_- y^{\lambda-1-1} + B_2 \lambda_- y^{\lambda-1-1} + \bar{\kappa}} \right), \]

\[ m < y : \quad \frac{\pi}{X} = -\frac{\theta}{\sigma} (\lambda - 1). \]

4.1 Properties of the Optimal Policy

Figure 4: (Benchmark) Optimal consumption (C) and optimal ratio of wealth in risky assets (\( \frac{\Pi}{X} \)) to wealth (X) when \( \gamma = 3, m = 2, \rho = 0.05, \mu = 0.07, r = 0.01, \sigma = 0.2 \)

Figure 4 shows optimal consumption behavior and the optimal ratio of wealth in risky assets for a Friedman-Savage consumer when \( \gamma = 3, \) and \( u(\bar{C}) = 2. \) We will denote this as the standard or benchmark Friedman-Savage case. Here \( u(\bar{C}) \) indicates the utility gained from consuming at \( \bar{C}. \) For simplicity, we let \( \bar{C} = 1. \) To highlight the distinctive behavior of the Friedman-Savage type consumer, we also plot the behavior of an agent with an unmodified CRRA felicity for comparison. In the plot depicting optimal consumption behavior, the dotted line corresponds to the consumption to wealth behavior of a CRRA utility maximizer, and the solid line with the jump corresponds to the consumption to wealth behavior of a Friedman-Savage utility maximizer. The figure indicates that the Friedman-Savage agent does not increase her consumption smoothly - for low levels of wealth she will con-
sume at a minimum subsistence level, and only when her wealth reaches a certain trigger value she will start consuming $\bar{C}$ – she will not consume any other level between 0 and $\bar{C}$. As her wealth gets even higher she increases her consumption monotonously, in a manner similar to that of an investor with normal CRRA utility.

For the optimal proportion of wealth in risky assets, the dotted horizontal line corresponds to the proportion invested in risky assets for a CRRA investor, and is found by $\pi = \frac{0}{\sigma \gamma}$. For these parameters it is equal to 0.5. The curved line corresponds to the ratio of wealth invested in risky assets for a Friedman-Savage consumer. Consistent with Theorem 2(4), we see that for a wealth level of zero until a certain threshold level, the optimal proportion in risky assets is invariably 3 – much higher than that of an investor with a normal CRRA utility function. This threshold is the wealth level that corresponds to the agent’s abrupt jump in consumption. After the threshold level the ratio of wealth invested in risky assets decreases. Thus, when the agent with Friedman-Savage type utility is using the portfolio to finance a constant, low level of consumption, she aggressively takes risk, but once her jump in consumption occurs, she decreases her risk taking. We see that as wealth increases the investor will gradually decrease her ratio of wealth in the risky asset until it converges to the ratio of a CRRA investor from above. Further, when $\gamma = 3$, and $m = 2$ we see that the Friedman Savage agent’s proportion of wealth in the risky asset will always be higher than or equal to that of a CRRA investor.
Figure 5 shows optimal consumption and proportion of investments in risky assets when $\gamma = 3$, $u(\tilde{C}) = 10$, $\rho = 0.05$, $\mu = 0.07$, $r = 0.01$, $\sigma = 0.2$.

Figure 5 shows optimal consumption and proportion of investments in risky assets when $\gamma = 3$, and $u(\tilde{C}) = 10$. We see that the agent’s consumption behavior is similar to the case when $u(\tilde{C}) = 2$. However we see that the agent will consume $\tilde{C}$ for larger levels of wealth than in the case of $u(\tilde{C})$. Since the utility gained from consuming at $\tilde{C}$ is high, it is natural to see the agent continue to consume $\tilde{C}$ for higher levels of wealth.

When the agent consumes at the subsistence consumption level, the optimal investment behavior is the same as the $u(\tilde{C}) = 2$ case. However, once the investor starts consuming at $\tilde{C}$, we see the investor decreases her proportion of wealth in risky assets much faster than when $u(\tilde{C}) = 2$. The utility gained from consuming at $\tilde{C}$ is higher than before, which means that once one is consuming at $\tilde{C}$ or higher, the utility loss when going back to the minimum consumption level is greater. Thus, we see that the agent becomes much more risk averse once she begins consuming $\tilde{C}$. Note that the proportion invested in risky assets is still above or equal to the CRRA level for all levels of wealth.
Figure 6: Optimal consumption (C) and optimal ratio of wealth in risky assets ($\frac{\Pi}{X}$) to wealth (X) when $\gamma = 0.9$, $u(\bar{C}) = 2$, $\rho = 0.05$, $\mu = 0.07$, $r = 0.01$, $\sigma = 0.2$

Figure 6 shows optimal consumption and investment in the risky asset when $\gamma = 0.9$. We see that the agent’s consumption behavior is similar to the case when $\gamma = 3$.

The investment behavior however shows interesting results. We see that when the agent’s wealth level is low, her ratio of wealth in the risky assets is the same as when $\gamma = 3$. However, once she reaches the wealth level that corresponds to the agent’s jump in consumption, the ratio of wealth invested in risky assets decreases sharply, and even falls under levels that would correspond to a normal CRRA investor. The ratio then increases monotonously with wealth and converges to the ratio of a normal CRRA investor from below. This implies that this Friedman-Savage investor would effectively become more risk averse for higher levels of wealth than a normal CRRA investor.

Figure 7 shows a sample path of optimal consumption and proportion of wealth in risky assets of a Friedman-Savage consumer when $\gamma = 3$, and $u(\bar{C}) = 2$. Figure 8 shows a sample path of optimal consumption and investment when $\gamma = 3$, and $u(\bar{C}) = 10$. The simulation period is 10 years, such that we can see some long-term behavior.

We see that when $u(\bar{C}) = 2$, the investor’s consumption will fall to the subsistence level quite often. However, when $u(\bar{C}) = 10$, the consumption will rarely fall to the subsistence level. We also see that the investor takes much more risk when
Figure 7: Sample path of consumption $C$ (left) and ratio of wealth in risky assets $\frac{\Pi_X}{\bar{X}}$ (right) when $\gamma = 3$, $u(\bar{C}) = 2$, $\rho = 0.05$, $\mu = 0.07$, $r = 0.01$, $\sigma = 0.2$

Figure 8: Sample path of consumption $C$ (left) and ratio of wealth in risky assets $\frac{\Pi_X}{\bar{X}}$ (right) when $\gamma = 3$, $u(\bar{C}) = 10$, $\rho = 0.05$, $\mu = 0.07$, $r = 0.01$, $\sigma = 0.2$
When $u(\bar{C}) = 10$, the proportion of investments in the risky asset is close to the CRRA level for quite a while, while when $u(\bar{C}) = 2$ the proportion invested in risky assets is usually above the CRRA level. We observe that when consumption falls to the subsistence level the investor effectively becomes much less risk averse and invests a high proportion in risky assets.

### 4.2 Comparative Statics

Figure 9: Optimal consumption ($C$) and ratio of wealth in risky assets ($\frac{\Pi}{X}$) to wealth ($X$) when $\gamma = 3, u(\bar{C}) = 2, \rho = 0.05, \mu = 0.02, r = 0.01, \sigma = 0.2$

Figure 9 shows the consumption and portfolio decisions when the market premium becomes smaller, in an attempt to examine if the almost gambling behavior of the agent exists. Consistent with Theorem 3(2), we see that when the market premium is small, the agent consumes at a level of zero for only a very short period and increases her consumption to $\bar{C}$ at a lower level of wealth compared to the benchmark case.

The agent’s investment behavior is also consistent with the results in Theorem 3(2). When the market premium is small, for very low levels of wealth, the Friedman-Savage consumer invests very aggressively compared to the benchmark above. Note here that the CRRA investor’s proportion invested in the risky asset is lower than the benchmark case. Further, we see quite an extreme difference in risk taking behavior between low wealth levels and higher wealth levels, and we see that the agent becomes very risk averse when her wealth is high enough to consume $\bar{C}$. 

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In Theorem 3(2) we noted that for small wealth levels the optimal ratio of wealth in risky assets declines as the risk premium increases, when the risk premium is close to 0. However, Figure 10 shows that this is not true when the risk premium is large. We see that when $\theta$ is high, both the CRRA investor and the Friedman-Savage investor invest a higher ratio of wealth into risky assets for all levels of wealth compared to the benchmark case. Note that once the Friedman-Savage agent’s wealth allows her to consume at $\bar{C}$, she decreases her proportion of wealth invested in risky assets, but not as abruptly as the case when $\theta$ is low.

Figure 10 also shows the consumption behavior for large $\theta$. When the market premium of risk is high, we see the agent’s consumption behavior is similar to the benchmark case, except that she jumps to the higher consumption level $\tilde{C}$ at a lower wealth level than the benchmark.
Figure 11: Optimal consumption (C) and ratio of wealth in risky assets (Π/X) to wealth (X) when \( \gamma = 3, u(\bar{C}) = 2, \rho = 0.01, \mu = 0.07, r = 0.01, \sigma = 0.2 \)

Figure 12: Optimal consumption (C) and ratio of wealth in risky assets (Π/X) to wealth (X) when \( \gamma = 3, u(\bar{C}) = 2, \rho = 0.09, \mu = 0.07, r = 0.01, \sigma = 0.2 \)

Figure 11 shows the consumption and portfolio decisions when the agent is patient. We see that she consumes at the minimum subsistence level for larger levels of wealth than usual. This is intuitive, since a patient investor values future consumption more than current consumption, she will be willing to endure low levels of consumption longer than a benchmark Friedman-Savage investor.

Figure 12 shows the consumption and portfolio decisions when the agent is impatient. Unsurprisingly, we see that the impatient consumer increases her consumption to the higher level \( \bar{C} \) at a lower level of wealth than the benchmark case.
Since current consumption is very important for her, she needs to consume at a higher level as early as possible.

Consistent with Proposition 1, we see that for wealth levels supporting the subsistence consumption level, as the Friedman-Savage consumer becomes more patient, her risk taking declines. One interesting thing to note is that once consuming at the higher wealth level, the impatient investor becomes very risk averse, while the patient investor decreases her proportion of wealth in the risky asset at a slower rate than the standard investor.

5 Conclusion

We have examined the consumption and portfolio decisions of an agent with Friedman-Savage type period utility in continuous time. Using the martingale method developed by Karatzas et al. (1987) and Cox and Huang (1989), we have found closed form solutions for the value function and the optimal consumption and investment policies.

Our main findings can be written as follows. First, in continuous time the investor with Friedman-Savage type utility does not gamble. This is because in our intertemporal model the value function is strictly concave throughout its whole region. This is consistent with Bailey et al. (1980), who showed that in the intertemporal case, because of risk and consumption smoothing, borrowing and saving can replicate or dominate the utility of gambling.

Second, the optimal consumption for the Friedman-Savage investor does not increase monotonously with increasing wealth. Rather, there will be a jump in consumption corresponding to the region where the utility is convex. Before the jump the investor will consume at a benchmark or subsistence level of consumption $C$, and after the jump the investor will consume at a certain level of consumption $\bar{C}$. Only after the wealth increases to the region where the utility of consumption is concave will the Friedman-Savage investor increase her consumption monotonously with wealth.

Nevertheless, we find that investors will aggressively invest in risky activities for the wealth levels that support only a minimum subsistence level of consumption. This confirms the findings of Vereshchagina and Hopenhayn (2009) that poorer
entrepreneurs will undertake more risk. We also verify the intuition of Friedman and Savage that “intermediate income groups⁷ might be expected to hold relatively large shares of their assets in moderately speculative common stocks and to furnish a disproportionate fraction of entrepreneurs.” (Friedman and Savage, 1948, p. 302).

However, as the premium of the risky asset approaches zero we observe that the agent’s investment in the risky asset approaches infinity, and the wealth level at which the agent adjusts her consumption to the right endpoint of the convex region approaches zero. We call this *almost gambling*. As their wealth increases, they decrease their holding of risky investments and in the limit their portion of wealth invested in risky assets will be equal to that of an investor with normal concave utility. The agent exhibits highly aggressive risk-taking behavior for some wealth levels and conservative behavior for large wealth levels. Thus, the model illustrates most of Friedman-Savage’s intuition even though gambling does not exist.

When wealth is supporting the subsistence level of consumption, the amount of risk taken by the Friedman-Savage consumer increases with her subjective discount factor $\rho$. That is, an impatient agent will invest more in the risky assets when she is consuming at the subsistence level of wealth. The patient investor, however, will invest less into the risky assets. These also confirm the findings of Vereshchagina and Hopenhayn (2009) that impatient entrepreneurs will be more willing to take on risky projects.

References


⁷ (author’s comments) those whose wealth will fall in the region where utility of wealth is convex.


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A Proof of Theorem 2

1. Define $\tilde{u} \equiv \max_c \tilde{u}(C(t)) - y(t)C(t) = \tilde{u}(C^*(t)) - y(t)C^*(t)$. It is a convex function and is called the convex dual of $\tilde{u}$. We consider the problem facing the agent at time $t \geq 0$.

Let us define a function $J(y(t))$ by the following

$$J(y(t)) = E\left[\int_t^\infty e^{-\rho(s-t)}\tilde{u}(y(t)) \, dt \bigg| \mathcal{F}_t \right].$$

That is, $J(y(t))$ is equal to the first part of the Lagrangian maximized over consumption. Then, by the Feynman-Kac Theorem (Karatzas and Shreve
(1991)) $J$ satisfies the following equation:

$$
\frac{1}{2} \theta^2 y^2 J''(y) + (\rho - r) y J'(y) - \rho J(y) + \tilde{u} = 0.
$$

A general solution to the equation can be written as the sum of a general solution to the homogeneous equation and a particular solution:

$$
J(y) = Ay^\lambda_- + By^\lambda_+ + J_p,
$$

where $A, B$ are constants and $\lambda_+, \lambda_-$ are the solutions to the characteristic equation

$$
\frac{1}{2} \theta^2 \lambda^2 + \left( \rho - r - \frac{1}{2} \theta^2 \right) \lambda - \rho = 0,
$$

i.e.,

$$
\lambda_+ = \frac{-(\rho - r - \frac{1}{2} \theta^2) + \sqrt{(\rho - r - \frac{1}{2} \theta^2)^2 + 2 \rho \theta^2}}{\theta^2},
\lambda_- = \frac{-(\rho - r - \frac{1}{2} \theta^2) - \sqrt{(\rho - r - \frac{1}{2} \theta^2)^2 + 2 \rho \theta^2}}{\theta^2}.
$$

For $y \leq l$, because of the transversality condition the coefficient $A$ should vanish. Similarly, for $y > m$, the coefficient $B$ should vanish. Thus, $J(y)$ takes the following forms in our problem

- $y \leq l$:
  
  $$
  J = B_1 y^{\lambda_+} + J_{p,\tilde{u}}(y),
  $$

- $l < y < m$:
  
  $$
  J = A_1 y^{\lambda_-} + B_2 y^{\lambda_+} + \frac{\tilde{c}}{r} y + \frac{1}{\rho} u(\tilde{c}),
  $$

- $m < y$:
  
  $$
  J = A_3 y^{\lambda_-},
  $$

where $J_{p,\tilde{u}}(y)$ is a particular solution to the equation for the case $\tilde{u}(y) = \tilde{u} \left( \tilde{u}^{-1} (y) \right) - y \tilde{u}^{-1} (y)$, which can be found by a standard method, e.g. the method of variation of parameters (see e.g., Boyce and DiPrima (2008)).

The unknown coefficients for $J$ can be found by the following smooth pasting conditions

$$
B_1 l^{\lambda_+} + J_{p,\tilde{u}}(l) = A_2 l^{\lambda_-} + B_2 l^{\lambda_+} - \frac{\tilde{c}}{r} l + \frac{1}{\rho} u(\tilde{c}),
$$

$$
A_2 m^{\lambda_-} + B_2 m^{\lambda_+} - \frac{\tilde{c}}{r} m + \frac{1}{\rho} u(\tilde{c}) = A_3 m^{\lambda_-},
$$
\[
\begin{align*}
\lambda + B_1 l^{\lambda - 1} + J'_p \tilde{u}(l) &= \lambda A_2 l^{\lambda - 1} + \lambda B_2 l^{\lambda - 1} - \frac{\bar{c}}{r}, \\
\lambda A_2 m^{\lambda - 1} + \lambda B_2 m^{\lambda - 1} - \frac{\bar{c}}{r} &= \lambda A_3 m^{\lambda - 1}.
\end{align*}
\]

The first two equations guarantee continuity of \( J \) and the last two equations guarantee smoothness (continuity of the derivative) of \( J \).

Finally, the value function \( V \) of the problem can be obtained by the following duality

\[
V(X(t)) = \min_{y(t)} (J(y(t)) + y(t)X(t)).
\]

From the above we can obtain \( J'(y(t)) = -X(t) \) and thus \( y = J^{-1}(-X) \). Therefore,

\[
V(X(t)) = \left( J(J^{-1}(-X)) + J^{-1}(-X)X \right).
\]

2. To derive the optimal consumption policy, the first order conditions of Problem C

\[
\max_{c} \bar{u}(C(t)) - y(t)C(t)
\]

are

\[
\begin{align*}
\bar{u}'(C) &= y & y &\leq l, \\
\bar{u}'(C) &\in Diff(\bar{C}) & l &< y \leq m, \\
\bar{u}'(C) &\in Diff(\bar{C}) & y &> m.
\end{align*}
\]

3. To derive the optimal portfolio, we inspect the wealth evolution equation implied by the indirect utility function \( J \) and compare it with (1). By Ito’s lemma, the implied wealth evolution equation is

\[
dX(t) = -((\rho - r)y(t)J''(y(t)) + \frac{1}{2}\theta^2 y(t)^2 J'''(y(t)))dt - \theta y(t)J''(y(t))dW(t).
\]

Comparing with (1) we can obtain the portfolio process \( \pi(t) = \frac{\theta y(t)J''(y(t))}{\sigma} \).
Similarly, we obtain the ratio of wealth invested in risky assets

\[
\frac{\pi(t)}{X(t)} = -\frac{\theta y(t) J''(y(t))}{\sigma J'(y(t))}.
\]

4. Verification that the policy derived in the above is optimal can be provided by Theorem 2.3, Cox and Huang (1989).

**B Proof of Theorem 3**

1. Let \( x \) and \( y \) be two wealth levels and \( \{c^*_x(t)\}_{t=0}^{\infty} \) and \( \{c^*_y(t)\}_{t=0}^{\infty} \) be optimal consumption streams corresponding to \( x \) and \( y \), respectively. Let \( s \in (0, 1) \). Then, \( \{sc^*_x(t) + (1-s)c^*_y(t)\}_{t=0}^{\infty} \) is a feasible consumption stream when initial wealth is equal to \( sx + (1-s)y \). But concavity of \( \bar{u}(c) \) implies that

\[
s\bar{u}(c^*_x(t)) + (1-s)\bar{u}(c^*_y(t)) \leq \bar{u}(sc^*_x(t) + (1-s)c^*_y(t)).
\]

Thus,

\[
sV(x) + (1-s)V(y) = E \int_0^\infty e^{-\rho t} (s\bar{u}(c^*_x(t)) + (1-s)\bar{u}(c^*_y(t))) \, dt \leq E \int_0^\infty e^{-\rho t} \bar{u}(sc^*_x(t) + (1-s)c^*_y(t)) \, dt \leq V(sx + (1-s)y).
\]

This shows that \( V \) is concave. Next we show its strict concavity. It will be sufficient that the first inequality in the above is strict. Note that the first inequality in the above is equality only when almost surely for almost every \( t \)

\[
s\bar{u}(c^*_x(t)) + (1-s)\bar{u}(c^*_y(t)) = \bar{u}(sc^*_x(t) + (1-s)c^*_y(t)),
\]

but this is true only when \( c^*_x(t) = 0 \) or \( \bar{c} \), and \( c^*_y(t) = 0 \) or \( \bar{c} \). Denoting \( y_x(t) \) the marginal utility wealth process when initial wealth is equal to 0, the prob-
ability of the event, $y_x(t) < l$ for every $t$ in a nonempty open time interval, is positive, since $y(t)$ follows a geometric Brownian motion (see e.g., Karlin and Taylor (1975)), and when $y_x(t) < l$, $e^{x(t)} > \bar{c}$. Thus, the above equality is violated in a set of a positive measure. This proves that $V$ is strictly concave.■

2. For $y > m$, we first show that $\lim_{\theta \to 0} \frac{\pi}{X} = \infty$

\[
\lim_{\theta \to 0} \frac{\pi}{X} = \lim_{\theta \to 0} - \frac{\theta (\lambda_- - 1)}{\sigma} = \lim_{\theta \to 0} \frac{(\rho - r - \frac{1}{2} \theta^2) + \sqrt{(\rho - r - \frac{1}{2} \theta^2)^2 + 2 \rho \theta^2 - \theta^2}}{\theta \sigma} = \lim_{\theta \to 0} \left[ \left( \frac{\rho - r}{\theta} - \frac{1}{2} \theta \right) + \sqrt{\left( \frac{\rho - r}{\theta} - \frac{1}{2} \theta \right)^2 + \left( \rho - \frac{\theta}{\theta^2} \right)^2} \right] = \infty.
\]

Next to show that $\lim_{\theta \to 0} X[m] = 0$, where $X[m]$ is the wealth level corresponding to $y = m$, we note that

\[
\lim_{\theta \to 0} \lambda_\pm = \lim_{\theta \to 0} \left( \rho - r - \frac{1}{2} \theta^2 \right) \pm \sqrt{\left( \rho - r - \frac{1}{2} \theta^2 \right)^2 + 2 \rho \theta^2} = \lim_{\theta \to 0} \left( \frac{\rho - r}{\theta^2} - \frac{1}{2} \theta \right) \pm \sqrt{\left( \frac{\rho - r}{\theta^2} - \frac{1}{2} \theta \right)^2 + \left( \rho - \frac{\theta}{\theta^2} \right)^2} \to \pm \infty.
\]

$X[m]$ is given by the following

\[
X[m] = -J'(m) = -\lambda_- A_3 m^{\lambda_- - 1} = \frac{\tilde{c}}{r} - B_2 \lambda_+ m^{\lambda_+ - 1} - A_2 \lambda_- m^{\lambda_- - 1}.
\]

And we know
\[
\lim_{\theta \to 0} |A_2| = \lim_{\theta \to 0} \left| \kappa \left( \frac{\gamma}{1-\gamma} + \frac{1}{\lambda_+} \right) + \frac{\bar{C}}{r} \left( 1 - \frac{1}{\lambda_+} \right) - \frac{1}{\rho} \frac{\bar{C}^{\gamma-r}}{1-\gamma} \right| < \infty.
\]

Thus,
\[
\lim_{\theta \to 0} A_2 \lambda_+ \lambda_- m^{\lambda_- - 1} = 0,
\]
\[
\lim_{\theta \to 0} B_2 \lambda_+ \lambda_- m^{\lambda_- - 1} = \lim_{\theta \to 0} \frac{(1 - \lambda_-) \bar{C} r m + \frac{\lambda_-}{\rho} u(\bar{C}) \lambda_+}{(\lambda_+ - \lambda_-) m^\lambda} m^{\lambda_- - 1} = \lim_{\theta \to 0} \frac{(1 - \lambda_-) \bar{C} m + \frac{\lambda_-}{\rho} u(\bar{C}) \lambda_+}{(\lambda_+ - \lambda_-)} m^{-(\lambda_+ - \lambda_-) - 1} \to 0.
\]

Thus, \(\lim_{\theta \to 0} X[m] = 0.\]

C Proof of Proposition 1

1. When \(y > m\), the change in the optimal proportion of wealth invested in risky assets with respect to change in the subjective discount factor is
\[
\frac{\partial \pi}{\partial \rho} = \frac{\partial}{\partial \rho} \left( \frac{\theta(1 - \lambda_-)}{\sigma} \right) = \frac{1}{\theta \sigma} \left( 1 + \frac{(\rho - r) + \frac{1}{2} \theta^2}{((\rho - r - \frac{1}{2} \theta^2) + 2\rho \theta^2)^{\frac{1}{2}}} \right) > 0,
\]
for all \(\rho > 0\).
\[ B_1 = \frac{\lambda_- A_2 l^{\lambda_- - 1} + \lambda_+ B_2 l^{\lambda_+ - 1} - \bar{C} l^{-\frac{1}{\gamma}} + \kappa l^{-\frac{1}{7}}}{\lambda_+ l^{\lambda_+ - 1}}, \]

\[ A_2 = \frac{\kappa l^{1-\frac{1}{7}} (\frac{\lambda_+ \gamma}{1-\gamma} + 1) + \bar{C} l^{1-\gamma} - \frac{\lambda_+ C^{1-\gamma}}{1-\gamma}}{l^{\lambda_+} (\lambda_+ - \lambda_-)}, \]

\[ B_2 = \frac{(1 - \lambda_-) \bar{C} m + \frac{\lambda_+}{\rho} u (\bar{C})}{(\lambda_+ - \lambda_-) m^{\lambda_+}}, \]

\[ A_3 = \frac{\bar{C} l^{-\frac{1}{7}} - B_2 \lambda_+ m^{\lambda_+ - 1} - \frac{A_2 \lambda_- m^{\lambda_- - 1}}{-\lambda_- m^{\lambda_- - 1}}}{}, \]